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Further Development of the Fekete-Szegö $\left|a_3-\mu a_2^2\right|$-Functional Inequality for Classes of Analytic Functions Based on Differential Operators and Subclasses

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Abstract

In mathematics, the Fekete-Szegö inequality is an inequality for the coefficients of univalent analytic functions found by Fekete and Szegö (1933), related to the Bieberbach conjecture. Finding similar estimates for other classes of functions is called the Fekete-Szegö problem. In this paper, I study to solve the Fekete-Szegö problem for $\left|a_3-\mu a_2^2\right|$-functional inequalities with μ is real or complex by the generalized linear differential operator. That is the main result in this paper.

Subject Areas

Mathematics

Keywords

Analytic Functions, Fekete-Szegö Problem, Subclass, Hadamard Product, Linear Operator, Strongly Starlike Functions, Strongly Convex Functions

1. Introduction

Suppose that $\mathbb{M}$ denote the class of all analytic $f(z)$ in the open unit disk

$$\Omega:=\left\{kz\in\mathbb{C}:|kz|<1,k\in\mathbb{N}\setminus\{0\}\right\}.$$

With

$$\mathbb{M}:=\left\{g\mid g(kz):=kz+\sum_{n=2}^{\infty}ka_nz^n,g:\Omega\to\mathbb{C}\right\}$$

A typical problem in geometric function theory is to study a functional made up of com-binations of the coefficients of the original function. Usually, there is a parameter over which the extremal value of the functional is needed. The paper deals with one important functional of this type: the Fekete-Szegö functional. The classical Fekete-Szegö functional is defined by

$$\Delta_{\mu}(f) = \left|a_3 - \mu a_2^2\right|, (0 < \mu < 1)$$

and it is derived from the Fekete-Szegö inequality. The problem of maximizing the abso-lute value of the functional μ in subclasses of normalized functions is called the Fekete-Szegö problem. In 1933, Fekete and Szegö [1] found the maximum value of $\left|a_3 - \mu a_2^2\right|$. as a function of the real parameters μ, for functions belonging to the class S. Since then, several researchers solved the Fekete-Szegö problem for various sublasses of the class of S and related subclasses of functions in A. The mathematicians who introduced the functional. M. Fekete and G. Szegö [1], were able to bound the classical functional in the class S by

$$1 + 2\exp\left\{\frac{-2\mu}{1-\mu}\right\} \geq 0.$$

Later Pfluger [2] used Jenkin's method to show that this result holds for complex μ such that $Re\left\{\frac{-2\mu}{1-\mu}\right\} \geq 0$. Keogh and Merkes [3] obtained the solution of the Fekete-Szegö problem for the class of close-to-convex functions. Ma and Minda [4] [5] gave a complete answer to the Fekete-Szegö problem for the classes of strongly close-to-convex functions and strongly starlike functions. In the literature, there exists a large number of results about inequalities for $\Delta_{\mu}(f)$ corresponding to various subclasses of A (see, for instance, [1]-[26]).

Suppose $\mathbb{E}$ is a subfamily of $\mathbb{M}$ consisting of functions that are univalent in Ω. For functions $f, h \in \mathbb{M}$, *i.e.* f, h is represented as:

$$f(kz) := kz + \sum_{n=2}^{\infty} ka_n z^n \tag{1}$$

and

$$h(lz) := lz + \sum_{n=2}^{\infty} lb_n z^n \tag{2}$$

I now define the Hadamard product of $f(z)$ and $h(z)$ as follows:

$$(f * h)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n = (h * f)(z), \forall z \in \Omega \tag{3}$$

Next I consider the linear operator L defined as follows

$$L(b,d) : \mathbb{M} \to \mathbb{M} :$$

$$L(b,d) \times f(kz) := G(b,d,z) * f(kz) = zk + \sum_{n=2}^{\infty} \frac{(b)_{n-1}}{(d)_{n-1}} kz^n, \forall z \in \Omega. \tag{4}$$

In there

$$b, d \in \mathbb{C} \setminus \{-1, -2, -3, \cdots\}.$$

$$G(b,d,z)=z+\sum_{n=2}^{\infty}\frac{(b)_{n-1}}{(d)_{n-1}}z^n,\forall z\in\Omega.$$

$(\eta)_n$ is the Pochhammer symbols defined, $\eta\in\mathbb{C}$ and in terms of the Euler Γ-function, by

$$(\eta)_n=\frac{\Gamma(\eta+n)}{\Gamma(\eta)}=\begin{cases}1 & \text{if } n=0,\\ \eta(\eta+1)\cdots(\eta+n-1) & \text{if } n\in\mathbb{N}^*,\end{cases} \tag{5}$$

The aim of this paper is to present Lemma 4 to construct an analytic D-function inequality for the Fekete-Szego problem using a different approach.

The paper is organized as follows:

In section preliminaries I remind some basic notations in [6]-[26] such as The generalized linear operator, the linear multiplier differential operator $D^m(\eta,\phi)f$, subclass $Q(m,\eta,\phi,\alpha,b,d)$.

Section: 3 Stability $\left|a_3-\mu a_2^2\right|$-functional inequalities for complex parameter μ.

Section: 4 Stability $\left|a_3-\mu a_2^2\right|$-functional inequalities for real parameter μ.

2. Preliminaries

Definition 1. The linear multiplier differential operator $D^m(\eta,\phi)f$ was defined as follows:

+ $D^0(\eta,\phi)f(z)=f(z)$,

+ $$D^1(\eta,\phi)f(z)=D(\eta,\phi)f(z)=\alpha\phi z^2(f(z))''+(\alpha-\phi)z(f(z))'+(1-\alpha+\phi)zf(z),$$

+ $D^2(\eta,\phi)f(z)=D(\eta,\phi)\left(D^1(\eta,\phi)f(z)\right)$,

⋮

+ $D^m(\eta,\phi)f(kz)=D(\eta,\phi)\left(D^{m-1}(\eta,\phi)f(kz)\right)$,

In there $\eta\ge\phi$ and $m\in\mathbb{N}$

From the definition I lead to consequence

Corollary 1. If $f\in\mathbb{M}$ then the linear multiplier differential operator $D^m(\eta,\phi)f$ identified as

$$D^m(\eta,\phi)f(kz)=kz+\sum_{n=2}^{\infty}\left[1+(\eta\phi n+\eta-\phi)(n-1)\right]^m a_n kz^n.$$

Definition 2. The generalized linear operator

$$L(m,\eta,\phi,b,d):\mathbb{M}\to\mathbb{M}$$

$$\begin{aligned}L(m,\eta,\phi,b,d)\times f(kz)&:=G(b,d,z)*D^m(\eta,\phi)f(kz)\\&=zk+\sum_{n=2}^{\infty}G_n^m(\eta,\phi)\frac{(b)_{n-1}}{(d)_{n-1}}ka_nz^n,\forall z\in\Omega\end{aligned}$$

In there $G_n^m(\eta,\phi)=\left[1+(\eta\phi n+\eta-\phi)(n-1)\right]^m$, $\eta\ge\phi\ge0$, $m\in\mathbb{N}$ and $b,d\ne-1,-2,-3,\cdots$.

Definition 3. Suppose that b,d be a nonzero complex numbers with $b,d\ne-1,-2,-3,\cdots$, $\alpha\ge\phi$ and $\mu\in\mathbb{C}$, $0\le\eta\le\beta$ and $m\in\mathbb{N}$. Let $0<\eta\le1$.

A function $f \in \mathbb{M}$ is given by the following form

$$f(kz) = kz + \sum_{i=2}^{\infty} k a_i z^i$$

is said to belong to subclass $f \in Q(m,\eta,\phi,\alpha,b,d)$ if:

$$\left|\arg\left(\frac{z\left(L(m,\eta,\phi,b,d)\, f(kz)\right)'}{L(m,\eta,\phi,b,d)\, f(kz)}\right)\right| < \frac{\pi}{2}\alpha, z \in \Omega. \tag{6}$$

Note: This class includes a variety of well-known subclasses of $\mathbb{M}$.

For example,

$$\mathbb{M}_1(0,\eta,\phi,b,b) = \left\{ z \in \mathbb{M} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\alpha, z \in \Omega \right\}$$

$$\mathbb{M}_1(0,\eta,\phi,2,1) = \left\{ f \in \mathbb{M} : \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha, z \in \Omega \right\}$$

3. Stability $\left|a_3 - \mu a_2^2\right|$-Functional Inequalities for Complex Parameter μ

Let $\mathbb{P}$ be the class of all analytic functions

$$\mathbb{P} := \left\{ q(z) \mid q(z) = 1 + \frac{c_1}{n} z + \frac{c_2}{n} z^2 + \cdots, z \in \Omega, n \in \mathbb{N}^* \; Re q(z) > 0 \right\}$$

3.1. Condition for Existence of $\left|a_3 - \mu a_2^2\right|$-Functional Inequalities

Lemma 1. If

$$q(z) = 1 + \frac{c_1}{n} z + \frac{c_2}{n} z^2 + \cdots, \forall j, n \in \mathbb{N}^* \text{ and } \forall q(z) \in \mathbb{P}. \tag{7}$$

Then

1) $\left|\frac{c_j}{n}\right| \le 2, j \ge 1,$

2) $\left|\frac{c_2}{n} - \frac{c_1^2}{2n^2}\right| \le 2 - \frac{|c_1|^2}{2n^2}.$

Lemma 2. Suppose that b, d be a nonzero complex numbers with $b, d \ne -1, -2, -3, \cdots$, $\alpha \ge \phi$ and $\mu \in \mathbb{C}$, $0 \le \eta \le \beta$ and $m \in \mathbb{N}$. If $f \in A(m,\eta,\phi,\alpha,b,d)$ of the form:

$$f(kz) = kz + \sum_{i=2}^{\infty} k a_i z^i \tag{8}$$

Then

$$i / |a_2| \ge \frac{2\alpha |d|}{k G_2^m(\eta,\phi) |b|} \tag{9}$$

$$ii/\left|a_3\right| \le \begin{cases} \dfrac{(2k+1)\alpha^2|d||d+1|}{k^3G_2^m(\eta,\phi)|b||b+1|} & \text{if } \alpha \le \dfrac{1}{2k+\alpha}, \\ \dfrac{\alpha|d||d+1|}{k^3G_3^m(\eta,\phi)|b||b+1|} & \text{if } \alpha \ge \dfrac{1}{2k+\alpha}, \end{cases}$$

Proof. we put $F(z) = L(m,\beta,\phi,b,d)f(z) = kz + k\lambda_2 z^2 + k\lambda_3 z^3 + \cdots$. Since

$$F(x) = \frac{F'(kz)}{F(kz)} = (q(z))^\alpha, q \in \mathbb{P} \tag{10}$$

on the other hand

$$\frac{z\left(k + k\lambda_2 z + k\lambda_3 z^2 + \cdots\right)}{kz + k\lambda_2 z^2 + k\lambda_3 z^3 + \cdots} = \left(1 + \frac{c_1}{k}z + \frac{c_2}{k}z^2 + \cdots\right)^\alpha \tag{11}$$

which implies the equality

$$\begin{aligned} & kz + k\lambda_2 z^2 + k\lambda_3 z^3 + \cdots \\ & = kz + (\alpha c_1 + k\lambda_2)z + \left(\alpha c_2 + \frac{\alpha(\alpha-1)}{2k}c_1^2 + \alpha\lambda_1 + k\lambda_3\right)z^3 \end{aligned}$$

Equating the coefficients of both sides we get

$$\lambda_2 = \frac{\alpha c_1}{k}, \lambda_3 = \frac{\alpha}{2}\left(\frac{c_2}{k} - \frac{\alpha c_1^2}{2k}\right) + \frac{2k+1}{4k^2}\alpha^2 c_1^2 \tag{12}$$

Therefore, according to (12) we have

$$\begin{aligned} F(z) = \varphi(b,d,z) \times D^m(\eta,\phi) f(zk) = kz + \sum_{n=2}^{\infty} G_n^m(\eta,\phi)\frac{(b)_{n-1}}{(d)_{n-1}}ka_n z^n \\ kz + \sum_{n=2}^{\infty} G_n^m(\eta,\phi)\frac{\Gamma(b+n-1)\Gamma(d)}{\Gamma(d+n-1)\Gamma(b)}ka_n z^n \end{aligned} \tag{13}$$

Thus we have

$$\frac{\alpha c_1}{k} = G_2^m(\eta,\phi)\frac{\Gamma(b+1)\Gamma(d)}{\Gamma(d+1)\Gamma(b)}ka_2 = G_2^m(\eta,\phi)\frac{b}{d}ka_2 \tag{14}$$

$$a_2 = \frac{\alpha d c_1}{k^2 b G_2^m(\eta,\phi)} \tag{15}$$

From Lemma 1, We have

$$|a_2| \le \frac{2\alpha|d|}{k|b|G_2^m(\eta,\phi)} \tag{16}$$

From (12) and (13), we get

$$\begin{aligned} & \frac{\alpha}{2}\left(\frac{c_2}{k} - \frac{\alpha c_1^2}{2k}\right) + \frac{2k+1}{4k^2}\alpha^2 c_1^2 \\ & = G_3^m(\eta,\phi)\frac{\Gamma(b+2)\Gamma(d)}{\Gamma(d+2)\Gamma(b)}ka_3 = G_3^m(\eta,\phi)\frac{b(b+1)}{d(d+1)}ka_3 \end{aligned} \tag{17}$$

So, We obtain

$$a_3 = \frac{\alpha}{2k}\frac{d(d+1)}{G_3^m(\eta,\phi)b(b+1)}\left(\frac{\alpha}{2}\left(\frac{c_2}{k}-\frac{\alpha c_1^2}{2k}\right)+\frac{2k+1}{4k^2}\alpha^2 c_1^2\right) \tag{18}$$

$$|a_3| \le \frac{\alpha}{2k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(b+1)|}\left\{2-\frac{|c_1|^2}{2k^2}++\frac{2k+1}{2k^2}\alpha|c_1^2|\right\} \tag{19}$$

$$|a_3| \le \frac{\alpha}{4k^3}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(b+1)|}\left\{4k^2-|c_1|^2+2(2k+1)\alpha|c_1|^2\right\} \tag{20}$$

3.2. Construct the of $\left|a_3-\mu a_2^2\right|$-Function Inequality

Theorem 1. Suppose that *a*, *c* be a complex parameters such that $b,d \ne -1,-2,-3,\cdots$, $\alpha \ge \phi$ and $\mu \in \mathbb{C}$, $0 \le \eta \le \beta$ and $m \in \mathbb{N}$. If $f \in A(m,\eta,\phi,\alpha,b,d)$, $\alpha \in (0,1)$ and μ is a complex parameter, then

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(b+1)|}\max\left\{\frac{\left(4k^2-1\right)G_2^{2m}(\eta,\phi)|b||(d+1)|+|\alpha Q(G,\mu,b,d)|}{4k^2 G_2^m(\eta,\phi)|b||(d+1)|}\right\} \tag{21}$$

In there

$$Q(G,\mu,b,d)=(2k+1)G_2^{2m}(\eta,\phi)b(d+1)+4k\mu G_3^m(\eta,\phi)d(b+1) \tag{22}$$

Proof. We have

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{d(d+1)}{G_3^m(\eta,\phi)b(b+1)}\left(\frac{\alpha}{2k^2}\left(c_2-\frac{\alpha c_1^2}{2k}\right)+\frac{2k+1}{4k^3}\alpha^2 c_1^2\right)-\mu\frac{\alpha^2 d^2 c_1^2}{k^2 b^2 G_2^{2m}(\eta,\phi)} \\ &= \frac{\alpha}{2k}\frac{d(d+1)}{G_3^m(\eta,\phi)b(b+1)}\left(\frac{c_2}{k}-\frac{c_1^2}{2k}\right)+\frac{2k+1}{4k^3}\alpha^2 c_1^2\frac{d(d+1)}{G_3^m(\eta,\phi)b(b+1)} \\ &\quad -\mu\frac{\alpha^2 d^2 c_1^2}{k^2 b^2 G_2^{2m}(\eta,\phi)} \\ &= \frac{\alpha}{2k}\frac{d(d+1)}{G_3^m(\eta,\phi)b(b+1)}\left(\frac{c_2}{k}-\frac{\alpha c_1^2}{2k}\right) \\ &\quad +\frac{\alpha^2 d\left[(2k+1)G_2^{2m}(\eta,\phi)b(d+1)-4k\mu G_3^m(\eta,\phi)d(b+1)\right]}{4k^3 G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)b^2(b+1)}c_1^2 \end{aligned} \tag{23}$$

Thus, we get

$$\begin{aligned} \left|a_3-\mu a_2^2\right| &\le \frac{\alpha}{2k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(b+1)|}\left|\frac{c_2}{k}-\frac{c_1^2}{2k}\right| \\ &\quad +\frac{\alpha^2|d|\left|(2k+1)G_2^{2m}(\eta,\phi)b(d+1)-4k\mu G_3^m(\eta,\phi)d(b+1)\right|}{4k^3 G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}\left|c_1^2\right| \end{aligned} \tag{24}$$

By Lemma 3.1 we have

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{2k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(b+1)|}\left|2-\frac{|c_1^2|}{2k}\right| + \frac{\alpha^2|d|\left|(2k+1)G_2^{2m}(\eta,\phi)b(d+1)-4k\mu G_3^m(\eta,\phi)d(b+1)\right|}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}|c_1^2| \tag{25}$$

So, I get

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(b+1)|}-\frac{|c_1^2|}{4k^3}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(b+1)|} + \frac{\alpha^2|d|\left|(2k+1)G_2^{2m}(\eta,\phi)b(d+1)-4k\mu G_3^m(\eta,\phi)d(b+1)\right|}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}|c_1^2| \tag{26}$$

So

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(b+1)|} + \frac{\alpha|d|\left[\alpha\left|(2k+1)G_2^{2m}(\eta,\phi)b(d+1)-4k\mu G_3^m(\eta,\phi)|d||(b+1)|\right|-G_2^{2m}(\eta,\phi)b(d+1)\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}|c_1^2| \tag{27}$$

Next by (22), I get

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(d+1)|} + \frac{\alpha|d|\left[\alpha|Q(G,\mu,b,d)|-\mu G_2^{2m}(\eta,\phi)|d||(b+1)|\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}|c_1^2| \tag{28}$$

Case 1: If

$$|Q(G,\mu,b,d)| \le \mu G_2^{2m}(\eta,\phi)|d||(b+1)|$$

Then from (28) I got the result

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(d+1)|} \tag{29}$$

Case 2: If

$$|Q(G,\mu,b,d)| \ge \mu G_2^{2m}(\eta,\phi)|d||(b+1)|$$

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &\le \frac{\alpha}{k}\frac{|d||(d+1)|}{G_3^m(\eta,\phi)|b||(d+1)|} + \frac{\alpha|d|\left[\alpha|Q(G,\mu,b,d)|-G_2^{2m}(\eta,\phi)|d||(b+1)|\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|} \\ &= \frac{4k^2G_2^{2m}\alpha|b||d||d+1|+|d|\left[\left|\alpha^2Q(G,\mu,b,d)\right|-\alpha G_2^{2m}(\eta,\phi)|d||(b+1)|\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|} \\ &= \frac{(4k^2+1)G_2^{2m}\alpha|b||d||d+1|+|d|\left[\left|\alpha^2Q(G,\mu,b,d)\right|\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}\end{aligned} \tag{30}$$

4. Construct the of $\left|a_3-\mu a_2^2\right|$-Function Inequality for Real Parmerte μ

Theorem 2. Suppose that $d,b\in(0,\infty)$, $\alpha\in(0,1]$, $\alpha\geq\phi\geq 0$ and $m\in\mathbb{N}$. If $f\in A(m,\eta,\phi,\alpha,b,d)$, and $f\in\mathbb{M}$ then for $\mu\in\mathbb{R}$ we have.

$$\left|x_3-\mu a_2^2\right|$$

$$\leq\begin{cases}\dfrac{\alpha^2(2k+1)d(d+1)G_2^{2m}(\eta,\phi)-4k\mu d(b+1)G_3^m(\eta,\phi)}{G_3^m(\eta,\phi)G_2^{2m}(\eta,\phi)b^2(b+1)} \\ \quad\text{if }\mu\leq\dfrac{(\alpha(2k+1))b(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha d(b+1)G_3^m(\eta,\phi)}, \\ \dfrac{\alpha d(d+1)G_3^m(\eta,\phi)}{b^2(b+1)} \\ \quad\text{if }\dfrac{(\alpha(2k+1)-1)d(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha dG_3^m(\eta,\phi)G_2^{2m}(\eta,\phi)b^2(b+1)}\leq\mu\leq\dfrac{(\alpha(2k+1)+1)d(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha dG_3^m(\eta,\phi)G_2^{2m}(\eta,\phi)b^2(b+1)}, \\ \dfrac{\alpha d(4k\mu+1)d(b+1)G_3^m(\eta,\phi)-(2k+1)b(d+1)G_2^{2m}(\eta,\phi)}{G_3^m(\eta,\phi)G_2^{2m}(\eta,\phi)b^2(b+1)} \\ \quad\text{if }\dfrac{(\alpha(2k+1)-1)d(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha dG_3^m(\eta,\phi)G_2^{2m}(\eta,\phi)b^2(b+1)},\end{cases}$$

Proof. To prove Theorem 3.4 I consider the following cases:

Case 1:

Suppose that

$$\frac{(\alpha(2k+1)-1)d(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha dG_3^m(\eta,\phi)G_2^{2m}(\eta,\phi)b^2(b+1)}\geq\mu$$

From (27) I have

$$\left|a_3-\mu a_2^2\right|\leq\frac{\alpha}{k}\frac{d(d+1)}{G_3^m(\eta,\phi)b(b+1)}+\frac{\alpha d\left[(\alpha(2k+1)-1)G_2^{2m}(\eta,\phi)b(d+1)-4k\mu G_3^m(\eta,\phi)d(b+1)\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)\left|b^2\right|\left|(b+1)\right|}\left|c_1^2\right| \tag{31}$$

By lemma 1 with $\left|c_1^2\right|\leq 2$ I get

$$\left|a_3-\mu a_2^2\right|\leq\frac{\alpha}{k}\frac{d(d+1)}{G_3^m(\eta,\phi)|b||(b+1)|}+\frac{\alpha d\left[(\alpha(2k+1)-1)G_2^{2m}(\eta,\phi)b(d+1)-4k\mu G_3^m(\eta,\phi)d(b+1)\right]}{kG_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)\left|b^2\right|\left|(b+1)\right|} \tag{32}$$

Now I prove the case 2

Case 2:

$$\frac{(\alpha(2k+1)-1)b(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha dG_3^m(\eta,\phi)G_2^{2m}(\eta,\phi)b^2(b+1)}\leq\mu$$

(31) I have

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{k}\frac{d(d+1)}{G_3^m(\eta,\phi)|b||(b+1)|} + \frac{\alpha d\left[4k\mu\alpha G_3^m d(b+1)-(\alpha(2k+1)-1)G_2^{2m}(\eta,\phi)b(d+1)\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}\left|c_1^2\right| \le \frac{\alpha}{k}\frac{d(d+1)}{G_3^m(\eta,\phi)|b||(b+1)|} \quad (33)$$

Next I prove the case 3

Case 3:

$$\mu \ge \frac{(2k+1)b(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha dG_3^m(\eta,\phi)}$$

in this case I put

$$Q(G,\mu,b,d)=4k\mu G_3^m(\eta,\phi)d(b+1)-(2k+1)G_2^{2m}(\eta,\phi)b(d+1) \quad (34)$$

Next from (27) I have

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{k}\frac{d(d+1)}{G_3^m(\eta,\phi)|b||(b+1)|} + \frac{\alpha d\left[4k\mu\alpha G_3^m d(b+1)-(\alpha(2k+1)-1)G_2^{2m}(\eta,\phi)b(d+1)\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}\left|c_1^2\right| \quad (35)$$

Next continue to apply Lemma 1, I get

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha^2 d\left[4k\mu G_3^m d(b+1)-(2k+1)G_2^{2m}(\eta,\phi)b(d+1)\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|} \quad (36)$$

Case 4: Continue to apply the above inequality, I have

$$\mu \le \frac{(2k+1)b(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha d(b+1)G_3^m(\eta,\phi)}$$

Form (35), I have

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{k}\frac{d(d+1)}{G_3^m(\eta,\phi)|b||(b+1)|} + \frac{\alpha d\left[4k\mu\alpha G_3^m d(b+1)-(\alpha(2k+1)-1)G_2^{2m}(\eta,\phi)b(d+1)\right]}{4k^3G_2^{2m}(\eta,\phi)G_3^m(\eta,\phi)|b^2||(b+1)|}\left|c_1^2\right| \le \frac{\alpha}{k}\frac{d(d+1)}{G_3^m(\eta,\phi)|b||(b+1)|} \quad (37)$$

So the complete theorem 4.1 proves.

Corollary 2. Suppose that $d,b\in(0,\infty)$, $\alpha\in(0,1]$, $\alpha\ge\phi\ge0$ and $m\in\mathbb{N}$, $\mu\in\mathbb{R}$ and

$$0<\alpha\le\frac{(2k+1)b(d+1)G_2^{2m}(\eta,\phi)}{4k\alpha d(b+1)G_3^m(\eta,\phi)}. \quad (38)$$

If $f \in A\left(m,\eta,\phi,\alpha,b,d\right)$,

$$f : \mathbb{U} \to \mathbb{C}$$

$$f\left(kz\right) = kz + \sum_{i=2}^{k} ka_i z^i \tag{39}$$

then

$$\left|a_3\right| - \left|a_3\right| \le \frac{\alpha d\left(d+1\right)}{4k\alpha b\left(b+1\right)G_3^m\left(\eta,\phi\right)} \tag{40}$$

5. Conclusion

In this article, I have presented Lemma 1 to prove the existence of functional inequalities involving complex and real parameters (of the $\left|a_3 - \mu a_2^2\right|$-functional inequalities for complex parameter μ and $\left|a_3 - \mu a_2^2\right|$-functional inequalities for real parameter μ).

Conflicts of Interest

The author declares no conflicts of interest.

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